Exact Implementation of Higher Order Bayliss–Turkel Absorbing Boundary Operators in Finite-Element Simulation

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***Abstract*—A simple yet powerful scheme is employed to incorporate Bayliss–Turkel absorbing boundary conditions (ABC's) of *any order* into the finite-element simulation of open-region radiation problems. Unlike previous attempts to apply higher order ABC's in finite elements, the new implementation is *exact.* The exact implementation is made possible by incorporating normal derivatives in the ABC formulation through direct algebraic substitution. This scheme is found to offer enhanced accuracy while negligibly affecting the sparsity of the system matrix.**

***Index Terms*— Absorbing boundary conditions, differential equations techniques, finite elements, numerical methods, wave propagation.**

I. Introduction

THE ABSORBING boundary conditions (ABC's) introduced in the late 1970's by Bayliss and Turkel were perhaps the first operators to provide a mechanism by which a finite-element mesh can be terminated without the necessity to include excessively large number of elements in the region surrounding the structure [1]. The Bayliss–Turkel (BT) operators became very popular because of their simplicity and good accuracy in comparison to the more primitive Sommerfeld radiation condition.

The BT operators promised increased accuracy as the order increased. Despite this attractive feature, however, only first- and second-order operators were actually implemented in finite-element simulation. The reason for this was primarily due to the common understanding that the implementation of analytic ABC's (of the BT type, the Engquist–Majda type or others) into the finite-element formulation required expressing the operator in the following form:

$$u_\rho = \alpha u + \beta u_{\phi\phi} + \gamma u_{\phi\phi\phi} \cdots \tag{1}$$

where α, β, and γ are constants that depend on the wave number and the location of the outer boundary [see [2] for discussion of BT and higher order operators having the form in (1)]. The form in (1) is characterized by the presence of the radial derivative on one side and tangential derivatives on the other.

Transforming and constraining any operator to the form in (1) was thought essential because it lead to a straightforward implementation in the analytic equations of finite elements. In the case of the two-dimensional (2-D) space, the normal derivative of the field would appear under the line integral of the outer boundary [2]. More explicitly, let us consider the homogeneous Helmholz equation in 2-D free space. Assuming scattering by a perfectly conducting object, Galerkin procedure leads to

$$\int_\Omega \nabla v \cdot \nabla u - k^2 v u \, d\Omega = \int_{\Gamma_c} v \frac{\partial u}{\partial n} \, d\Gamma + \int_{\Gamma_o} v \frac{\partial u}{\partial n} \, d\Gamma \tag{2}$$

where Ω is the solution space of the problem and Γ_c and Γ_o are the conductor boundary and the cylindrical mesh-termination (outer) boundary, respectively. The unknown field is represented by u and the testing function by v.

Observation of (2) shows that an ABC having the form in (1) can be easily substituted in the integral over Γ_o. The alternative to (1) would be to use a from (actual expression or an approximation of the ABC) that would involve mixed derivatives.

Unfortunately, only the first- and second-order BT operators can be cast in the form given in (1). BT operators of third or higher order contain radial derivatives of second or higher orders, or might contain mixed derivatives. The implementation of these higher order operators would unfortunately add additional complexity into the finite-element formulation and necessitates the use of higher order finite elements which would then reduce the sparsity of the system matrix.

Despite this, however, one can find in the literature several ABC constructions based on the BT operators that extend beyond the second order. These constructions, however, are only approximations of higher order operators where mixed derivatives are either assumed negligible or replaced by simple approximations that give results with varying degrees of accuracy. Unfortunately, such approximations, for the most part, are unsubstantiated analytically and often result in unsatisfactory levels of accuracy.

In this work, we depart from the above constraints, namely that of recasting the boundary operator into the classical format in (1). Instead of interchanging normal derivatives with tangential derivatives, we maintain the original form of the operator and express (discretize) higher order operators as telescopic finite-difference expressions. The additional attractive features of this implementation are 1) it does not require the use of higher order finite elements and 2) it is remarkably simple to

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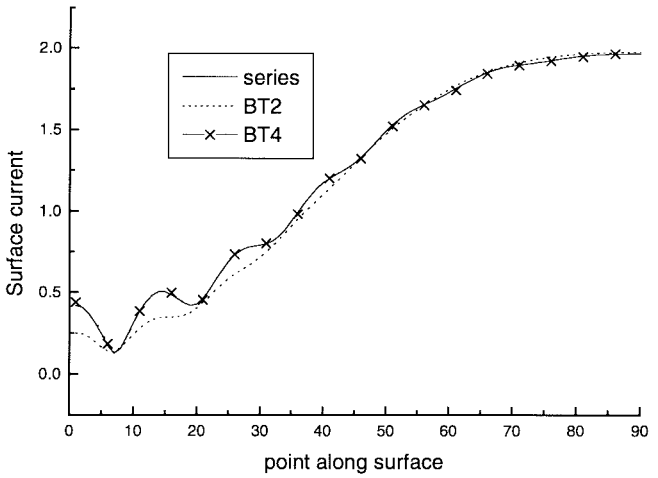

Fig. 1. Surface current for TE-polarization calculated using BT second-order (BT2), BT fourth-order (BT4), and the series solution. The x-axis represents equally spaced points spanning one half of the cylinder's surface.

implement, while affecting the overall sparsity of the finite element matrix in a negligible way.

The discussion in this work will be focused on the scalar 2-D wave equation, as would be the case in TE- or TM-polarization scattering. Extension to three-dimensional (3-D) scalar and vector wave equations is differed to future papers.

II. Formulation

The Nth-order BT operator applied on a cylindrical outer boundary is expressed as [1]

$$B\{u\} = \prod_{i=1}^{N} \left(\frac{\partial}{\partial \rho} + \frac{2i - 3/2}{\rho} + jk \right) = 0 \tag{3}$$

where ρ is the radial variable and k is the wave number. The form in (3) can be transformed into a discrete operator using the backward finite-difference approximation for the normal derivative, vis.

$$\partial_\rho \rightarrow \frac{(I - S^{-1})}{\Delta \rho} \tag{4}$$

where I is the identity operator and S^{-1} is the space shift operator. Substituting (4) into (3), we arrive at the finite-difference representation of the boundary operator in (3)

$$B^d\{u\} = \prod_{i=1}^{N} \left(\alpha_i I - \frac{S^{-1}}{\Delta \rho} \right) = 0 \tag{5}$$

where $\alpha_i = 1/\Delta\rho + jk + (2i - 3/2)/\rho$. In (5), B^d denotes a discrete operator. Notice that (5) is in effect a one-sided (using nodes internal to the boundary) finite-difference transformation of the operator in (3). Such transformation of the derivative was used successfully in the implementation of higher order Higdon boundary operators in the finite-difference time-domain (FDTD) method [3].

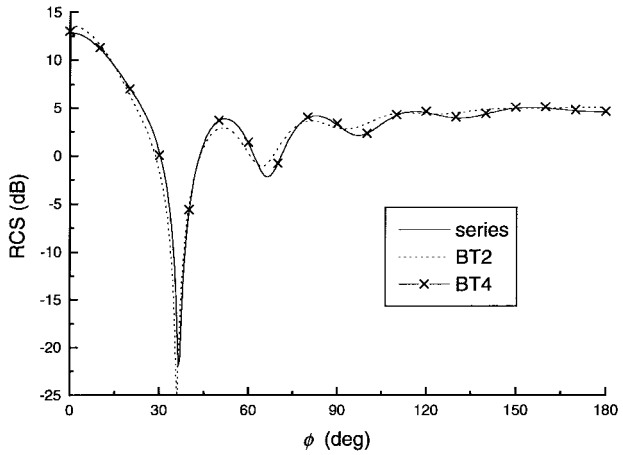

Fig. 2. Radar cross section (RCS) for 1-λ radius cylinder.

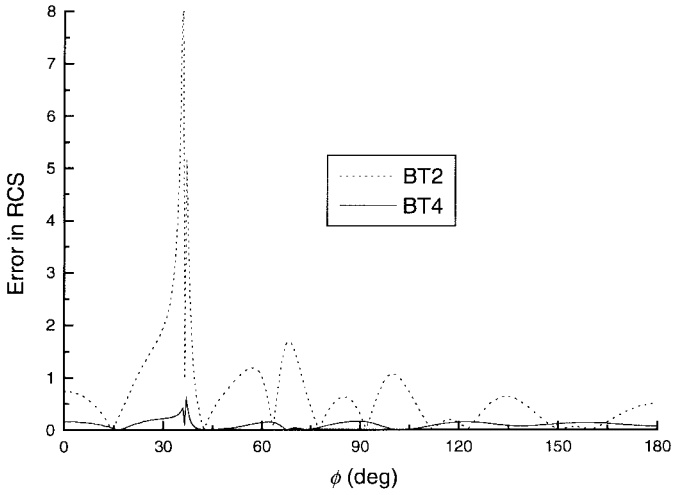

Fig. 3. Error in the RCS.

III. Implementation into Finite-Elements Simulation

The implementation of (5) into a finite-element formulation is carried out by direct substitution into the finite element matrix. For brevity, we skip the finite element formulation and start with the matrix equation. We split the unknown nodal field vector $\mathbf{U}$ into two vectors: $\mathbf{U}_b$ and $\mathbf{U}_i$, denoting the boundary nodes and interior nodes, respectively. The boundary nodes are those that lie on the terminal outer boundary. Thus, as a representative matrix system, we have

$$\begin{bmatrix} \mathbf{M}_{bb} & \mathbf{M}_{bi} \\ \mathbf{M}_{ib} & \mathbf{M}_{ii} \end{bmatrix} \begin{bmatrix} \mathbf{U}_b \\ \mathbf{U}_i \end{bmatrix} = \begin{bmatrix} \mathbf{F}_b \\ \mathbf{F}_i \end{bmatrix}. \tag{6}$$

The vectors $\mathbf{F}_i$ and $\mathbf{F}_b$ are related to the excitation and all boundary conditions. Next, we express the boundary condition (5) in the following matrix equation:

$$\mathbf{U}_b = \mathbf{B}\mathbf{U}_i. \tag{7}$$

Here, the matrix $\mathbf{B}$ represents the discrete boundary operator. Finally, we substitute (7) in (6) to obtain the reduced system matrix

$$(\mathbf{M}_{ib}\mathbf{B} + \mathbf{M}_{ii})\mathbf{U}_i = \mathbf{F}_i. \tag{8}$$

The system in (8) is well posed in the sense that the nodal fields that lie directly interior to the outer boundary nodes are governed by the wave equation and are constrained by (5).

IV. Numerical Experiment

To test the effectiveness of this new and highly simple scheme, we consider a classical problem where the analytical solution is available for comparison. We consider a perfectly conducting circular cylinder of 1-λ radius. We then apply the second- and fourth-order BT operators at a distance of $1/10\lambda$ from the cylinder's surface. The grid for this problem consists of 180 uniformly spaced nodes in the angular direction and 6 uniformly spaced nodes in the radial direction.

Fig. 1 shows the surface current for TE-polarization incidence (ϕ is measured 180° from the direction of incidence). Figs. 2 and 3 show radar cross-section (RCS) calculations. These graphs demonstrate that the fourth-order BT operator gives a significant enhancement in accuracy. In fact, Fig. 3 shows that the absolute maximum error in RCS can be reduced below 0.5 dB, even within the null zone, which is a significant reduction from the solution incorporating the second-order BT operator.

V. Summary

This letter presented a method by which the BT operators of *any order* can be implemented in a finite-element scheme. The implementation does not involve any simplification of the operator and is therefore *exact.* The numerical implementation of this scheme is simple and it only requires simple algebraic manipulations. The resultant matrix equation's sparsity is reduced in a negligible manner. Any incremental increase in computational cost due to the change in sparsity is reduced, however, by solving a smaller matrix.

References

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